Using Progressions to Understand the Common Core Fraction Standards

Presenters: Jennifer Hagman & Zachary Hagman

MaThink 2015 February 7, 2015

Our Focus

Through activities and discussion, participants will gain a better understanding of the Common Core Progressions in general. More specifically, participants will delve into the progressions for grades 3-5 on fractions and grades 6-7 on ratio and proportional reasoning to see the purpose of the progressions and how they will be of great use to a teacher as she/he plans lessons. In addition, uses for a number line as a tool when teaching these concepts will be discussed. Questions from the participants regarding the CC Progressions will be answered and resources highlighted.

Progressions

The Common Core State Standards in mathematics were built on progressions, that is, narrative documents describing the progression of a topic across a number of grade levels, informed both by research on children's cognitive development and by the logical structure of mathematics. These documents trace major concepts through the grade levels. They explain which standards build upon one another, point out cognitive difficulties and pedagogical solutions, and give more detail on particularly knotty areas of the mathematics.

From: http://ime.math.arizona.edu/progressions/

Progressions

K-12 Domains

K	1	2	3	4	5	6	7	8
Geometry								
Measurement and Data						Statistics and Probability		
Number and Operations in Base Ten						The Number System		
Operations and Algebraic Thinking						Expressions and Equations		
Counting and Cardinality			Number and Operations---Fractions			Ratios and Proportional Relationships		Functions

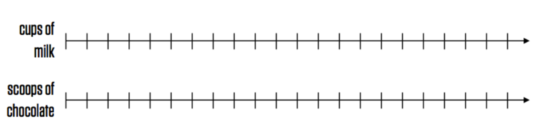
High School Conceptual Categories

NUMBER AND QUANTITY	ALGEBRA	FUNCTIONS	GEOMETRY	STATISTICS AND PROBABILITY

From: http://www.illustrativemathematics.org/

A Use for Number Lines

- How could we use a number line to solve the previous problem?

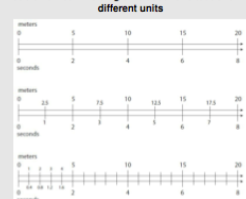


Relation to the Progressions

"Although it is traditional to move students quickly to solving proportions by setting up an equation, the Standards do not require this method in Grade 6" (pg 6).

"The study of proportional relationships is a foundation for the study of functions, which continues through High School and beyond" (pg 11).

Double number line diagrams used for situations with different units

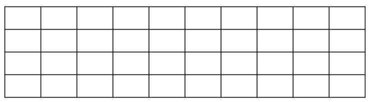


Double number lines indicate coordinated multiplying and dividing of quantities. This can also be indicated in tables.

From Progression 6-7, Ratios and Proportional Reasoning

Another problem.

- Consider the 4 by 10 grid below. Shade any 6 units.
- Use the picture to find the percent of the grid that is shaded.

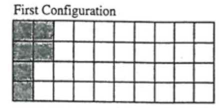


The Case of Ron Castleman

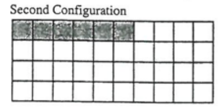
Some possible solutions:

- What percent does each column represent?
- What percent does each row represent?

First Configuration



Second Configuration

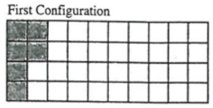


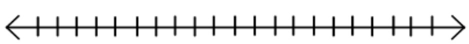
From: Implementing Standards-based Mathematics Instruction: A Casebook for Professional Development, pgs 37-63

The Case of Ron Castleman

How can a number line be used?

First Configuration





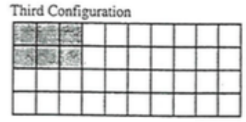
From: Implementing Standards-based Mathematics Instruction: A Casebook for Professional Development, pgs 37-63

The Case of Ron Castleman

Another possible solution:

- Describe how this student could have solved the problem.

Third Configuration



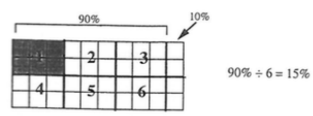


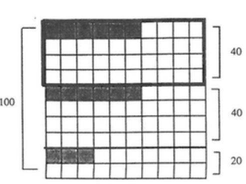
FIGURE 5.4. The diagram displayed by Omar and Marcus.

From: Implementing Standards-based Mathematics Instruction: A Casebook for Professional Development, pgs 37-63

The Case of Ron Castleman

Yet, another possible solution:

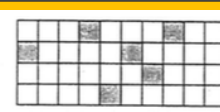
- What did this student do?
- Compare and contrast this method with the others we have discussed.



From: Implementing Standards-based Mathematics Instruction: A Casebook for Professional Development, pgs 37-63

The Case of Ron Castleman

Two girls were explaining how they solved using the figure at the right:



6 x 2.5% = 15%

Since there are 40 squares in the diagram and the whole diagram needed to be 100%, each small square represents 2 ½ percent.

After the girls shared, Michael asked "How could something that wasn't subdivided into 100 equal 100%?"

Derrick added, "Are you saying that 100%=1? I thought 100%=100!"

From: Implementing Standards-based Mathematics Instruction: A Casebook for Professional Development, pgs 37-63

Grade 3

The meaning of fractions In Grades 1 and 2, students use fraction language to describe partitions of shapes into equal shares.[2.G.3] In Grade 3 they start to develop the idea of a fraction more formally, building on the idea of partitioning a whole into equal parts. The whole can be a shape such as a circle or rectangle, a line segment, or any one finite entity susceptible to subdivision and measurement. In Grade 4, this is extended to include wholes that are collections of objects.

[2.G.3] Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words *halves*, *thirds*, *half of*, *a third of*, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.

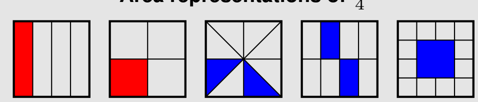
Grade 3 students start with unit fractions (fractions with numerator 1), which are formed by partitioning a whole into equal parts and taking one part, e.g., if a whole is partitioned into 4 equal parts then each part is $\frac{1}{4}$ of the whole, and 4 copies of that part make the whole. Next, students build fractions from unit fractions, seeing the numerator 3 of $\frac{3}{4}$ as saying that $\frac{3}{4}$ is the quantity you get by putting 3 of the $\frac{1}{4}$'s together.[3.NF.1] They read any fraction this way, and in particular there is no need to introduce "proper fractions" and "improper fractions" initially; $\frac{5}{3}$ is the quantity you get by combining 5 parts together when the whole is divided into 3 equal parts.

[3.NF.1] Understand a fraction $1/b$ as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size $1/b$.

The importance of specifying the whole

Without specifying the whole it is not reasonable to ask what fraction is represented by the shaded area. If the left square is the whole, the shaded area represents the fraction $\frac{3}{2}$; if the entire rectangle is the whole, the shaded area represents $\frac{3}{4}$.

Area representations of $\frac{1}{4}$



In each representation the square is the whole. The two squares on the left are divided into four parts that have the same size and shape, and so the same area. In the three squares on the right, the shaded area is $\frac{1}{4}$ of the whole area, even though it is not easily seen as one part in a division of the square into four parts of the same shape and size.

Two important aspects of fractions provide opportunities for the mathematical practice of attending to precision (MP6):

- Specifying the whole.
- Explaining what is meant by "equal parts."

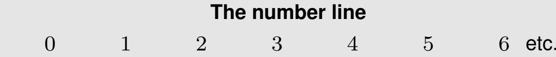
Initially, students can use an intuitive notion of congruence ("same size and same shape") to explain why the parts are equal, e.g., when they divide a square into four equal squares or four equal rectangles.

Students come to understand a more precise meaning for "equal parts" as "parts with equal measurements." For example, when a ruler is partitioned into halves or quarters of an inch, they see that each subdivision has the same length. In area models they reason about the area of a shaded region to decide what fraction of the whole it represents (MP3).

The goal is for students to see unit fractions as the basic building blocks of fractions, in the same sense that the number 1 is the basic building block of the whole numbers; just as every whole number is obtained by combining a sufficient number of 1s, every fraction is obtained by combining a sufficient number of unit fractions.

The number line and number line diagrams On the number line, the whole is the *unit interval*, that is, the interval from 0 to 1, measured by length. Iterating this whole to the right marks off the whole numbers, so that the intervals between consecutive whole numbers, from 0 to 1, 1 to 2, 2 to 3, etc., are all of the same length, as shown. Students might think of the number line as an infinite ruler.

The number line



To construct a unit fraction on a number line diagram, e.g. $\frac{1}{3}$, students partition the unit interval into 3 intervals of equal length

Multiplying and dividing fractions In Grade 4 students connected fractions with addition and multiplication, understanding that

$$\frac{5}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 5 \times \frac{1}{3}.$$

In Grade 5, they connect fractions with division, understanding that

$$5 \div 3 = \frac{5}{3},$$

or, more generally, $\frac{a}{b} = a \div b$ for whole numbers a and b, with b not equal to zero.[5.NF.3] They can explain this by working with their understanding of division as equal sharing (see figure in margin). They also create story contexts to represent problems involving division of whole numbers. For example, they see that

> If 9 people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get?

can be solved in two ways. First, they might partition each pound among the 9 people, so that each person gets $50 \times \frac{1}{9} = \frac{50}{9}$ pounds. Second, they might use the equation $9 \times 5 = 45$ to see that each person can be given 5 pounds, with 5 pounds remaining. Partitioning the remainder gives $5\frac{5}{9}$ pounds for each person.

Students have, since Grade 1, been using language such as "third of" to describe one part when a whole is partitioned into three parts. With their new understanding of the connection between fractions and division, students now see that $\frac{5}{3}$ is one third of 5, which leads to the meaning of multiplication by a unit fraction:

$$\frac{1}{3} \times 5 = \frac{5}{3}.$$

This in turn extends to multiplication of any quantity by a fraction.[5.NF.4a] Just as

$$\frac{1}{3} \times 5 \text{ is one part when } 5 \text{ is partitioned into } 3 \text{ parts,}$$

so

$$\frac{4}{3} \times 5 \text{ is } 4 \text{ parts when } 5 \text{ is partitioned into } 3 \text{ parts.}$$

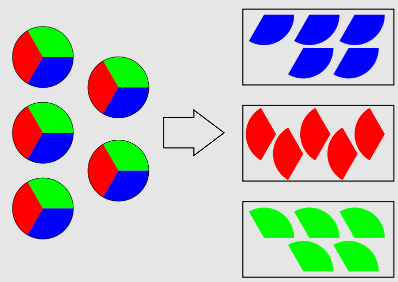
Using this understanding of multiplication by a fraction, students develop the general formula for the product of two fractions,

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd},$$

for whole numbers a, b, c, d, with b, d not zero. Grade 5 students need not express the formula in this general algebraic form, but rather reason out many examples using fraction strips and number line diagrams.

5.NF.3 Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

How to share 5 objects equally among 3 shares:
$5 \div 3 = 5 \times \frac{1}{3} = \frac{5}{3}$

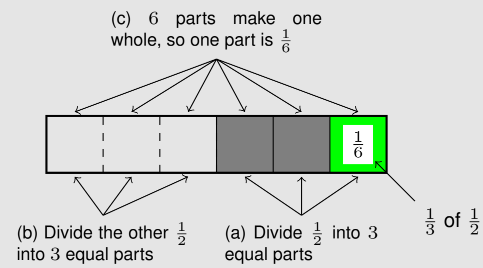


If you divide 5 objects equally among 3 shares, each of the 5 objects should contribute $\frac{1}{3}$ of itself to each share. Thus each share consists of 5 pieces, each of which is $\frac{1}{3}$ of an object, and so each share is $5 \times \frac{1}{3} = \frac{5}{3}$ of an object.

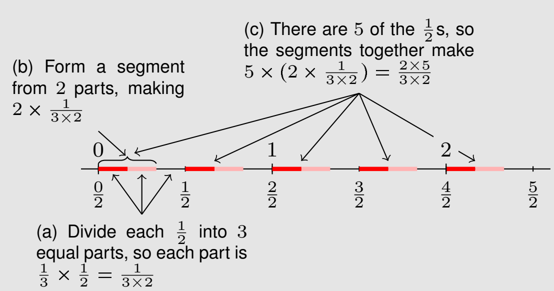
5.NF.4a Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

a Interpret the product $(a/b) \times q$ as a parts of a partition of q into b equal parts; equivalently, as the result of a sequence of operations $a \times q \div b$.

Using a fraction strip to show that $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$



Using a number line to show that $\frac{2}{3} \times \frac{5}{2} = \frac{2\times5}{3\times2}$



Grade 6

Representing and reasoning about ratios and collections of equivalent ratios Because the multiplication table is familiar to sixth graders, situations that give rise to columns or rows of a multiplication table can provide good initial contexts when ratios and proportional relationships are introduced. Pairs of quantities in equivalent ratios arising from whole number measurements such as "3 lemons for every \$1" or "for every 5 cups grape juice, mix in 2 cups peach juice" lend themselves to being recorded in a table.[6.RP.3a] Initially, when students make tables of quantities in equivalent ratios, they may focus only on iterating the related quantities by repeated addition to generate equivalent ratios.

As students work with tables of quantities in equivalent ratios (also called ratio tables), they should practice using and understanding ratio and rate language.[6.RP.1,6.RP.2] It is important for students to focus on the meaning of the terms "for every," "for each," "for each 1," and "per" because these equivalent ways of stating ratios and rates are at the heart of understanding the structure in these tables, providing a foundation for learning about proportional relationships in Grade 7.

Students graph the pairs of values displayed in ratio tables on coordinate axes. The graph of such a collection of equivalent ratios lies on a line through the origin, and the pattern of increases in the table can be seen in the graph as coordinated horizontal and vertical increases.[6.EE.9]

6.RP.3a Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.

a Make tables of equivalent ratios relating quantities with whole-number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios.

6.RP.1 Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities.

6.RP.2 Understand the concept of a unit rate a/b associated with a ratio $a : b$ with $b \neq 0$, and use rate language in the context of a ratio relationship.

6.EE.9 Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation.

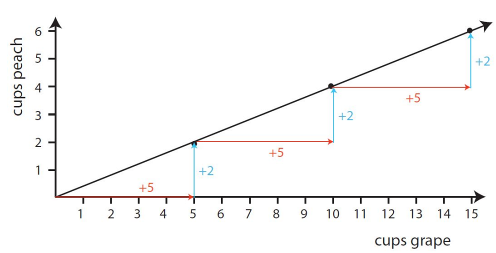
Showing structure in tables and graphs

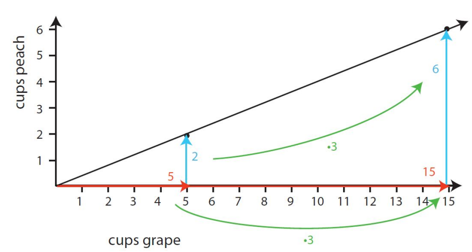
Additive Structure

cups grape	cups peach
5	2
10	4
15	6
20	8
25	10

Multiplicative Structure

cups grape	cups peach
5	2
10	4
15	6
20	8
100	40





In the tables, equivalent ratios are generated by repeated addition (left) and by scalar multiplication (right). Students might be asked to identify and explain correspondences between each table and the graph beneath it (MP1).

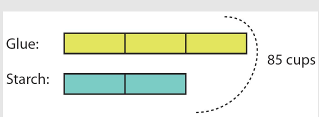
these other entries will require the explicit use of multiplication and division, not just repeated addition or skip counting. For example, if Seth runs 5 meters every 2 seconds, then Seth will run 2.5 meters in 1 second because in half the time he will go half as far. In other words, when the elapsed time is divided by 2, the distance traveled should also be divided by 2. More generally, if the elapsed time is multiplied (or divided) by N, the distance traveled should also be multiplied (or divided) by N. Double number lines can be useful in representing ratios that involve fractions and decimals.

As students become comfortable with fractional and decimal entries in tables of quantities in equivalent ratios, they should learn to appreciate that unit rates are especially useful for finding entries. A unit rate gives the number of units of one quantity per 1 unit of the other quantity. The amount for N units of the other quantity is then found by multiplying by N. Once students feel comfortable doing so, they may wish to work with abbreviated tables instead of working with long tables that have many values. The most abbreviated tables consist of only two columns or two rows; solving a proportion is a matter of finding one unknown entry in the table.

Measurement conversion provides other opportunities for students to use relationships given by unit rates.[6.RP.3d] For example, recognizing "12 inches in a foot," "1000 grams in a kilogram," or "one kilometer is $\frac{5}{8}$ of a mile" as rates, can help to connect concepts and methods developed for other contexts with measurement conversion.

Representing a problem with a tape diagram

Slimy Gloopy mixture is made by mixing glue and liquid laundry starch in a ratio of 3 to 2. How much glue and how much starch is needed to make 85 cups of Slimy Gloopy mixture?



$$
\begin{aligned}
5 \text{ parts} &\longrightarrow 85 \text{ cups}\\
1 \text{ part} &\longrightarrow 85 \div 5 = 17 \text{ cups}\\
3 \text{ parts} &\longrightarrow 3 \cdot 17 = 51 \text{ cups}\\
2 \text{ parts} &\longrightarrow 2 \cdot 17 = 34 \text{ cups}
\end{aligned}
$$

51 cups glue and 34 cups starch are needed.

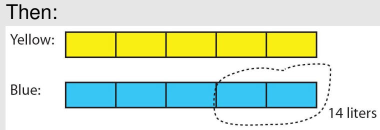
Tape diagrams can be useful aids for solving problems.

Representing a multi-step problem with two pairs of tape diagrams

Yellow and blue paint were mixed in a ratio of 5 to 3 to make green paint. After 14 liters of blue paint were added, the amount of yellow and blue paint in the mixture was equal. How much green paint was in the mixture at first?

At first:

Then:



$$
\begin{aligned}
2 \text{ parts} &\longrightarrow 14 \text{ liters}\\
1 \text{ part} &\longrightarrow 14 \div 2 = 7 \text{ liters}\\
\text{(original total) } 8 \text{ parts} &\longrightarrow 8 \cdot 7 = 56 \text{ liters}
\end{aligned}
$$

There was 56 liters of green paint to start with.

This problem can be very challenging for sixth or seventh graders.

A progression of strategies for solving a proportion

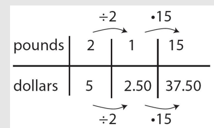
If 2 pounds of beans cost \$5, how much will 15 pounds of beans cost?

Method 1

pounds	2	4	6	8	10	12	14	1	15
dollars	5	10	15	20	25	30	35	2.50	37.50

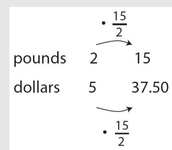
"I found 14 pounds costs \$35 and then 1 more pound is another \$2.50, so that makes \$37.50 in all."

Method 2



"I found 1 pound first because if I know how much it costs for each pound then I can find any number of pounds by multiplying."

Method 3



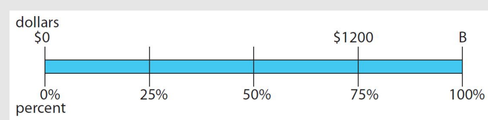
The previous method, done in one step.

With this perspective, the second column is seen as the first column times a number. To solve the proportion one first finds this number.

6.RP.3d Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities.

Solving a percent problem

If 75% of the budget is \$1200, what is the full budget?



"I said 75% is 3 parts and is \$1200
25% is 1 part and is $\$1200 \div 3 = \$400$
100% is 4 parts and is $4 \cdot \$400 = \$1600$"

portion	75	3	1200
whole	100	4	1600

75% is $\frac{1200}{B}$ 75% of B is 1200

$\frac{75}{100} = \frac{1200}{B}$ $\frac{75}{100} \cdot B = 1200$

$B = 1600$

In reasoning about and solving percent problems, students can use a variety of strategies. Representations such as this, which is a blend between a tape diagram and a double number line diagram, can support sense-making and reasoning about percent.

Connection to Geometry One new context for proportions at Grade 7 is scale drawings.[7.G.1] To compute unknown lengths from known lengths, students can set up proportions in tables or equations, or they can reason about how lengths compare multiplicatively. Students can use two kinds of multiplicative comparisons. They can apply a scale factor that relates lengths in two different figures, or they can consider the ratio of two lengths within one figure, find a multiplicative relationship between those lengths, and apply that relationship to the ratio of the corresponding lengths in the other figure. When working with areas, students should be aware that areas do not scale by the same factor that relates lengths. (Areas scale by the square of the scale factor that relates lengths, if area is measured in the unit of measurement derived from that used for length.)

Connection to Statistics and Probability Another new context for proportions at Grade 7 is to drawing inferences about a population from a random sample.[7.SP.1] Because random samples can be expected to be approximately representative of the full population, one can imagine selecting many samples of that same size until the full population is exhausted, each with approximately the same characteristics. Therefore the ratio of the size of a portion having a certain characteristic to the size of the whole should be approximately the same for the sample as for the full population.

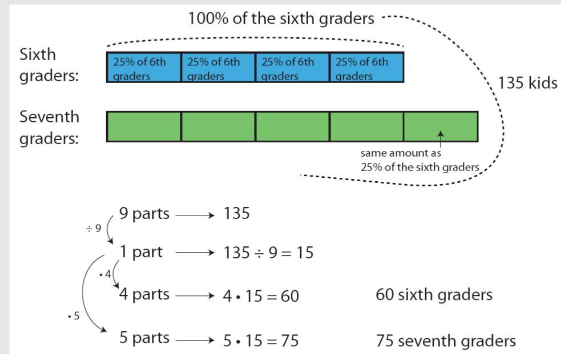
Where the Ratios and Proportional Relationships Progression is heading

The study of proportional relationships is a foundation for the study of functions, which continues through High School and beyond. Linear functions are characterized by having a constant rate of change (the change in the outputs is a constant multiple of the change in the corresponding inputs). Proportional relationships are a major type of linear function; they are those linear functions that have a positive rate of change and take 0 to 0.

Students extend their understanding of quantity. They write rates concisely in terms of derived units such as mi/hr rather than expressing them in terms such as "$\frac{3}{2}$ miles in every 1 hour." They encounter a wider variety of derived units and situations in which they must conceive units that measure attributes of interest.

Using percentages in comparisons

There are 25% more seventh graders than sixth graders in the after-school club. If there are 135 sixth and seventh graders altogether in the after-school club, how many are sixth graders and how many are seventh graders?



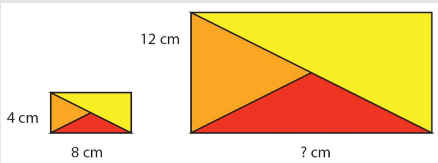
"25% more seventh graders than sixth graders means that the number of extra seventh graders is the same as 25% of the sixth graders."

[7.G.1] Solve problems involving scale drawings of geometric figures, including computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.

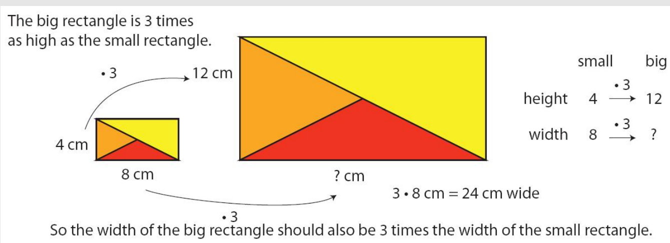
[7.SP.1] Understand that statistics can be used to gain information about a population by examining a sample of the population; generalizations about a population from a sample are valid only if the sample is representative of that population. Understand that random sampling tends to produce representative samples and support valid inferences.

Connection to geometry

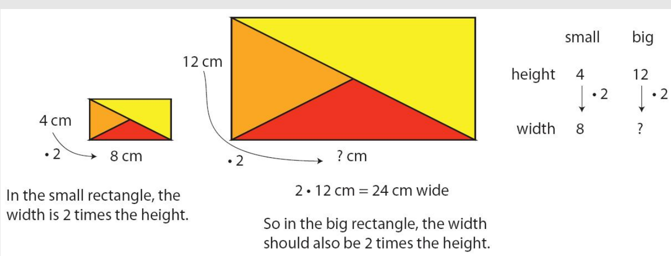
If the two rectangles are similar, then how wide is the larger rectangle?



Use a scale factor: Find the scale factor from the small rectangle to the larger one:



Use an internal comparison: Compare the width to the height in the small rectangle. The ratio of the width to height is the same in the large rectangle.



Connection to statistics and probability

There are 150 tiles in a bin. Some of the tiles are blue and the rest are yellow. A random sample of 10 tiles was selected. Of the 10 tiles, 3 were yellow and 7 were blue. What are the best estimates for how many blue tiles are in the bin and how many yellow tiles are in the bin?

Student 1

yellow:	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45
blue:	7	14	21	28	35	42	49	56	63	70	77	84	91	98	105
total:	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150

"I figured if you keep picking out samples of 10 they should all be about the same, so I got this ratio table. Out of 150 tiles, about 45 should be yellow and about 105 should be blue."

Student 2

yellow:	3	45
blue:	7	105
total:	10	150
	• 15	

"I also made a ratio table. I said that if there are 15 times as many tiles in the bin as in the sample, then there should be about 15 times as many yellow tiles and 15 times as many blue tiles. $15 \cdot 3 = 45$, so 45 yellow tiles. $15 \cdot 7 = 105$, so 105 blue tiles."

Student 3

30% yellow tiles	$30\% \cdot 150 = \frac{3 \cdot 10}{10 \cdot 10} \cdot 150 = \frac{3}{10} \cdot 15 \cdot 10 = 45$
70% blue tiles	$70\% \cdot 150 = \frac{7 \cdot 10}{10 \cdot 10} \cdot 150 = \frac{7}{10} \cdot 15 \cdot 10 = 105$

"I used percentages. 3 out of 10 is 30% yellow and 7 out of 10 is 70% blue. The percentages in the whole bin should be about the same as the percentages in the sample."